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On the Three-Point Couplings in Toda Field Theory

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ABSTRACT

Correlation functions of Toda field vertices are investigated by applying the method of integrating zero-mode developed for Liouville theory. We generalize the relations among the zero-, two- and three-point couplings known in Liouville case to arbitrary Toda theories. Two- and three-point functions of Toda vertices associated with the simple roots are obtained.

Since the impressive success of Goulian and Li [1] that the three-point correlators of the Liouville vertex functions appearing as the dressing factor of minimal model can be obtained by continuing the central charge to a certain value where the functional integral can be carried out exactly via free field technique [2], several authors have pushed forward their method [3, 4, 5] and generalized to define arbitrary three-point couplings in Liouville theory [4, 6, 7]. Since Liouville theory is a Toda field theory associated with the Lie algebra sl_2 and as two dimensional field theory Toda theories possess distinguished properties such as the W symmetry [8, 9, 10], it is natural to ask whether the similar functional integral approach can apply for the general Toda theories.

In this note we shall investigate correlators of the vertex functions of Toda fields by applying the method of ref. [2, 1]. We see that the zero-modes of the Toda fields can be integrated as in Liouville case and the remaining functional integrations over nonzero-modes can be carried out via free field technique if all the s -parameters are nonnegative integers. We thus arrive at complicated multiple integrals over complex planes. Unlike the Liouville case, we can not give closed forms of the integrals for three-point couplings. This prevent us from getting explicit expressions for general three-point couplings. We find, however, some universality among the three-point couplings for the vertex functions associated to simple roots. The relations among the zero-, two- and three-point couplings found in Liouville case [6] can also be generalized to Toda theory. In particular, we will give closed expressions for two- and three-point functions of the vertices associated to simple roots.

We consider the Toda field theory associated with the simple Lie algebra \mathcal{G} of rank r . The Toda field φ is an r -component vector in the root space. Let us denote the simple roots by α^a ($a = 1, \dots, r$), then it is described by the classical field equations

$$\partial_z \partial_{\bar{z}} \varphi - \frac{\mu^2}{8} \sum_{a=1}^r \alpha^a e^{\alpha^a \cdot \varphi} = 0 . \quad (1)$$

For $\mathcal{G} = sl_2$ this reduces to the Liouville equation.

The equation of motion (1) is invariant under the conformal reparametrization $z \rightarrow \xi = f(z)$ by the shift

$$\varphi \rightarrow \varphi - \rho \ln \partial_z f \partial_{\bar{z}} \bar{f} , \quad (2)$$

where ρ is a vector in the root space satisfying

$$\rho \cdot \alpha^a = 1 , \quad (3)$$

for any simple root. In terms of the fundamental weights λ^a ($a = 1, \dots, r$) defined by $2\lambda^a \cdot \alpha^b / (\alpha^a)^2 = \delta^{ab}$, it is given by

$$\rho = \sum_{a=1}^r \frac{2\lambda^a}{(\alpha^a)^2} \quad (4)$$

In quantum theory we start from the action

$$S[\hat{g}; \varphi] = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left(\hat{g}^{\alpha\beta} \partial_\alpha \varphi \cdot \partial_\beta \varphi + Q \cdot \varphi \hat{R} + \frac{\mu^2}{\gamma^2} \sum_{a=1}^r e^{\gamma \alpha^a \cdot \varphi} \right), \quad (5)$$

where $\hat{g}_{\alpha\beta}$ is the fiducial metric on the Riemann surface. We have introduced couplings Q , a vector in the root space, with the curvature and γ in the Toda potential. They are determined by requiring the conformal invariance.

The central charge of the Toda theory can be found most easily by applying the DDK's argument [11]. We note that the stress tensor for $\mu^2 = 0$ and in locally flat coordinates is given by

$$T(z) = -\frac{1}{2}(\partial_z \varphi)^2 + \frac{1}{2}Q \cdot \partial_z^2 \varphi \quad (6)$$

Using the operator product relation $\varphi_j(z)\varphi_k(w) \sim \delta_{jk} \ln |z-w|^2$, one can easily obtain

$$T(z)T(w) \sim \frac{1}{2} \frac{r + 3Q^2}{(z-w)^4}. \quad (7)$$

The central charge of the Toda sector is thus given by

$$c_\varphi = r + 3Q^2. \quad (8)$$

One can also determine the conformal dimension of arbitrary Toda vertex functions. Let β be an arbitrary vector in the root space, then the conformal dimension of the vertex function $e^{\beta \cdot \varphi}$ is given by

$$\Delta(e^{\beta \cdot \varphi}) = \frac{1}{2} \beta \cdot (Q - \beta). \quad (9)$$

In particular the Toda potential in (5) must be a (1,1)-conformal field by the requirement of conformal invariance. This relates Q with the coupling constant γ as

$$\gamma \alpha^a \cdot (Q - \gamma \alpha^a) = 2 \quad (10)$$

for any simple root. This is the quantum version of the classical relation (3) and has already been obtained in ref. [12]. Since the r simple roots are linearly independent each other, one can uniquely find the expression for Q as

$$Q = 2 \left(\frac{1}{\gamma} \rho + \gamma \bar{\rho} \right), \quad (11)$$

where $\bar{\rho}$ is simply the sum of all fundamental weights, *i.e.*,

$$\bar{\rho} = \sum_{a=1}^r \lambda^a . \quad (12)$$

For simply-laced Lie algebra one has $\bar{\rho} = \rho$. Hence, the renormalization of the couplings Q is simply a rescaling of the classical value as observed in Liouville theory. Since simple roots with different lengths coexist for nonsimply-laced cases, the couplings Q can not be proportional to ρ as can be easily seen from (10). Using (12), one can parametrize the central charge (8) in terms of γ . The Virasoro central charge in quantum Toda theories has been obtained in ref. [13]. A complete list for any simple Lie algebra is given in ref.[12].

Let us denote by c_{gh} and c_M the central charges of the ghost and matter sectors which are conformally coupled to the Toda fields. Then the total central charge must vanish to ensure the conformal invariance

$$c_\varphi + c_M + c_{gh} = 0 . \quad (13)$$

From (8), (10) and (11) we obtain the coupling constant as

$$\gamma = \sqrt{\frac{-c_{gh} - c_M - r + 24(|\rho||\bar{\rho}| - \rho \cdot \bar{\rho})}{48|\bar{\rho}|^2}} - \sqrt{\frac{-c_{gh} - c_M - r - 24(|\rho||\bar{\rho}| + \rho \cdot \bar{\rho})}{48|\bar{\rho}|^2}} , \quad (14)$$

where we have chosen a branch of the square roots so that the correct classical limit is recovered for $c_M \rightarrow -\infty$. Up to trivial rescaling of γ , this reproduces the well-known results for Liouville theory [11]. Due to the extended conformal symmetries [9] we must include ghosts with higher conformal dimensions for general Toda theories. By requiring γ to be real we find the bound for the matter central charge

$$c_M \leq -c_{gh} - r - 24(|\rho||\bar{\rho}| + \rho \cdot \bar{\rho}) \quad (15)$$

To illustrate in a concrete example consider A_n Toda theory. The holomorphic ghost sector of this system consists of pairs of ghost and anti-ghost with conformal weights $(-j, 0)$ and $(j+1, 0)$ for $j = 1, \dots, n$, each having central charge $-2(6j^2 + 6j + 1)$. The total ghost central charge is then given by

$$c_{gh} = -2n(2(n+1)^2 + 2(n+1) + 1) . \quad (16)$$

On the other hand the squared norm of ρ is easily found to be

$$\rho^2 = \frac{1}{12}n(n+1)(n+2) . \quad (17)$$

Putting these into (15) and noting $\bar{\rho} = \rho$, we obtain

$$c_M \leq n . \quad (18)$$

This generalizes the well-known $c = 1$ barrier for Liouville gravity to A_n Toda gravity.

We now turn to an arbitrary N -point functions of Toda vertices on the surface of spherical topology. It is defined by

$$G_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) = \int \mathcal{D}_{\hat{g}} \varphi e^{-S[\hat{g}, \varphi]} \prod_{j=1}^N e^{\beta_j \cdot \varphi(z_j)} . \quad (19)$$

We decompose the Toda field into the zero-mode φ_0 and the nonzero-mode $\tilde{\varphi}$ by

$$\varphi = \varphi_0 + \tilde{\varphi} , \quad \varphi_0 = \frac{\int d^2 z \sqrt{\hat{g}} \varphi}{\int d^2 z \sqrt{\hat{g}}} . \quad (20)$$

Then the integrations over the zero-mode can be carried out as in the Liouville case. We thus arrive at the functional integral

$$G_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) = |\det \alpha|^{-1} \prod_{a=1}^r \left(\frac{\mu^2}{8\pi\gamma^2} \right)^{s_N^a} \frac{\Gamma(-s_N^a)}{\gamma} I_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) , \quad (21)$$

where α is an $r \times r$ matrix with α^a as the a -th row elements and s_N^a is defined by

$$s_N^a = \frac{2\lambda^a \cdot \left(Q - \sum_{j=1}^N \beta_j \right)}{\gamma(\alpha^a)^2} . \quad (22)$$

The N -point function I_N in the rhs of (21) is given by

$$I_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) = \int \mathcal{D}_{\hat{g}} \tilde{\varphi} e^{-S_0[\hat{g}; \tilde{\varphi}]} \prod_{a=1}^r \left(\int d^2 z \sqrt{\hat{g}} e^{\gamma \alpha^a \cdot \tilde{\varphi}} \right)^{s_N^a} \prod_{j=1}^N e^{\beta_j \cdot \tilde{\varphi}(z_j)} . \quad (23)$$

where $S_0[\hat{g}; \tilde{\varphi}]$ is the action of r free massless bosons.

In general s_N^a take generic values for arbitrary β 's and the functional integration (23) can not be carried out explicitly. We avoid this difficulty by considering all the s_N^a being nonnegative integers and then continuing them back to the original generic values. In the case of Liouville theory s can be made a nonnegative integer by adjusting the central charge of the matter sector. Furthermore, continuation back to generic s is known by explicit construction [6, 7]. As for the Toda theory, all the s_N^a , in general, can not be made nonnegative integers at the same time by varying c_M . We can achieve this by adjusting one of the β 's for $N \geq 1$. In the following argument, we simply assume that the continuation back to arbitrary s_N^a exists. We

shall see, however, that the existence of such continuation procedure is not so essential in the present work. It is only necessary to show explicitly the conformality of (23) under Möbius transformations.

We choose the fiducial metric $\hat{g}_{\alpha\beta}$ to be everywhere flat except at infinity on the complex plane. In this case one can perform the functional integration as

$$\begin{aligned}
I_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) &= \int \mathcal{D}_{\hat{g}} \tilde{\varphi} e^{-S_0[\hat{g}; \tilde{\varphi}]} \int \prod_{a=1}^r \prod_{I=1}^{s_N^a} d^2 w_I^a \sqrt{\hat{g}(w_I^a)} e^{\gamma \alpha^a \cdot \tilde{\varphi}(w_I^a)} \prod_{j=1}^N e^{\beta_j \cdot \tilde{\varphi}(z_j)} \\
&= \int \prod_{a=1}^r \prod_{I=1}^{s_N^a} d^2 w_I^a \prod_{i < j}^N |z_i - z_j|^{-2\beta_i \cdot \beta_j} \prod_{i=1}^N \prod_{a=1}^r \prod_{I=1}^{s_N^a} |z_i - w_I^a|^{-2\gamma \beta_i \cdot \alpha^a} \\
&\quad \times \prod_{a < b}^N \prod_{I=1}^{s_N^a} \prod_{J=1}^{s_N^b} |w_I^a - w_J^b|^{-2\gamma^2 \alpha^a \cdot \alpha^b} \prod_{a=1}^r \prod_{I < J}^{s_N^a} |w_I^a - w_J^a|^{-2\gamma^2 (\alpha^a)^2} \quad (24)
\end{aligned}$$

Using the integral representation, one can check explicitly the conformal property of (23) under the Möbius transformations

$$z = \frac{a\xi + b}{c\xi + d} \quad (ad - bc = 1, \quad a, b, c, d \in \mathbf{C}). \quad (25)$$

We see that I_N is transformed as

$$I_N(z_1, \dots, z_N | \beta_1, \dots, \beta_N) = \prod_{i=1}^N |c\xi_i + d|^{4\Delta_i} I_N(\xi_1, \dots, \xi_N | \beta_1, \dots, \beta_N), \quad (26)$$

where Δ_j is the conformal dimension of the vertex function $e^{\beta_j \cdot \varphi}$. In showing this result, use has been made of (10) and (22).

We can not determine the z -dependence of the N -point function only from (26) for $N \geq 4$. For 3-point function, we can extract the usual conformal structure by fixing $\xi_1 = 0$, $\xi_2 = 1$, $\xi_3 = \infty$ as

$$I_3(z_1, z_2, z_3 | \beta_1, \beta_2, \beta_3) = \frac{C_3(\beta_1, \beta_2, \beta_3)}{|z_{12}|^{2\Delta_{12}} |z_{23}|^{2\Delta_{23}} |z_{31}|^{2\Delta_{31}}}, \quad (27)$$

where we have introduced $z_{ij} = z_i - z_j$ and $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ for $i, j, k = 1, 2, 3$ (cyclic), and $C_3(\beta_1, \beta_2, \beta_3)$ is defined by

$$\begin{aligned}
C_3(\beta_1, \beta_2, \beta_3) &= \lim_{\xi_3 \rightarrow \infty} |\xi_3|^{4\Delta_3} I_3(0, 1, \xi_3 | \beta_1, \beta_2, \beta_3) \\
&= \int \prod_{a=1}^r \prod_{I=1}^{s_3^a} d^2 w_I^a |w_I^a|^{-2\gamma \beta_1 \cdot \alpha^a} |1 - w_I^a|^{-2\gamma \beta_2 \cdot \alpha^a} \\
&\quad \times \prod_{a=1}^r \prod_{I < J}^{s_3^a} |w_I^a - w_J^a|^{-2\gamma^2 (\alpha^a)^2} \prod_{a < b}^r \prod_{I=1}^{s_3^a} \prod_{J=1}^{s_3^b} |w_I^a - w_J^b|^{-2\gamma^2 \alpha^a \cdot \alpha^b} \quad (28)
\end{aligned}$$

For the Liouville case the rhs of this expression reduces to the integral investigated by Dotsenko and Fateev [14] and be carried out explicitly. Furthermore, the resulting expression for C_3 can be continued to arbitrary β 's [6, 7]. Unfortunately, the closed form of the integral is not known to the present authors for Toda theories with $r \geq 2$ and we can not give explicit extension of the three-point coupling proposed in refs. [6, 7]. Below we will investigate the relations among the zero-, two- and three-point couplings by assuming the existence of $C_3(\beta_1, \beta_2, \beta_3)$ for arbitrary β 's that coincides to (28) if all the s_3^a are nonnegative integers. We will find some nontrivial constraints satisfied by the three-point couplings, which arise for $r > 1$ and have not appeared in the previous studies for the Liouville theory.

We define the three-point coupling $A_3(\beta_1, \beta_2, \beta_3)$ by

$$G_3(z_1, z_2, z_3 | \beta_1, \beta_2, \beta_3) = \frac{A_3(\beta_1, \beta_2, \beta_3)}{|z_{12}|^{2\Delta_{12}} |z_{23}|^{2\Delta_{23}} |z_{31}|^{2\Delta_{31}}} . \quad (29)$$

From (21) and (27), the three-point coupling A_3 can be written as

$$A_3(\beta_1, \beta_2, \beta_3) = |\det \alpha|^{-1} C_3(\beta_1, \beta_2, \beta_3) \prod_{a=1}^r \left(\frac{\mu^2}{8\pi\gamma^2} \right)^{s_3^a} \frac{\Gamma(-s_3^a)}{\gamma} . \quad (30)$$

The two-point function

$$G_2(z_1, z_2 | \beta, \beta) = \frac{1}{V_2} \int \mathcal{D}_{\hat{g}} \varphi e^{-S[\hat{g}; \varphi]} e^{\beta \cdot \varphi(z_1)} e^{\beta \cdot \varphi(z_2)} \quad (31)$$

can be obtained from the three-point function by the formula [6]

$$G_2(z_1, z_2 | \beta, \beta) = -\frac{\mu^2}{8\pi\gamma^2 s_2^a} \frac{1}{V_2} \int d^2 z_3 G_3(z_1, z_2, z_3 | \beta, \beta, \gamma \alpha^a) \quad (32)$$

where $\gamma s_2^a = 2\lambda^a \cdot (Q - 2\beta)/(\alpha^a)^2$ and α^a is an arbitrary simple root. V_2 is the infinite volume of the subgroup of the Möbius transformations (25) that leave the two points z_1 and z_2 fixed. It is given by

$$V_2 = \int \frac{d^2 z_3 |z_{12}|^2}{|z_{23}|^2 |z_{31}|^2} . \quad (33)$$

The same volume factor will factorize from the integral (32) and be canceled by the prefactor. We thus obtain the two-point function

$$\begin{aligned} G_2(z_1, z_2 | \beta, \beta) &= \frac{A_2(\beta)}{|z_{12}|^{4\Delta_\beta}} , \\ A_2(\beta) &= |\det \alpha|^{-1} C_3(\beta, \beta, \gamma \alpha^a) \prod_{b=1}^r \left(\frac{\mu^2}{8\pi\gamma^2} \right)^{s_2^b} \frac{\Gamma(-s_2^b)}{\gamma} , \end{aligned} \quad (34)$$

where $\Delta_\beta = \beta \cdot (Q - \beta)/2$. Using the general expression for the three-point coupling (30), we obtain

$$A_2(\beta) = -\frac{\mu^2}{8\pi\gamma^2 s_2^a} A_3(\beta, \beta, \gamma\alpha^a) \quad (a = 1, \dots, r). \quad (35)$$

This relation has already been noted for Liouville theory in refs. [6]. In the case of Toda theory new situation arises. Since we may choose one of the simple roots arbitrarily, (34) implies that $C_3(\beta, \beta, \gamma\alpha^a)$ is independent of the simple roots α^a .

The partition function A_0 can be obtained from the three-point function in the manner similar to the two-point function. Let $\alpha^{a,b,c}$ be arbitrary simple roots, then the partition function is given by

$$\begin{aligned} A_0 &= |\det \alpha|^{-1} C_3(\gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c) \prod_{d=1}^r \left(\frac{\mu^2}{8\pi\gamma^2} \right)^{s_0^d} \frac{\Gamma(-s_0^d)}{\gamma} \\ &= \left(\frac{\mu^2}{8\pi\gamma^2} \right)^3 \prod_{d=1}^r \frac{\Gamma(-s_0^d)}{\Gamma(-s_3^d)} A_3(\gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c), \end{aligned} \quad (36)$$

where $\gamma s_0^d = 2\lambda^d \cdot Q / (\alpha^d)^2$ and $s_3^d = s_0^d - \delta^{ad} - \delta^{bd} - \delta^{cd}$. The immediate consequence of (36) is that $C_3(\gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c)$ is independent of the combinations of simple roots. Since the dependences on the simple roots only arise from the Γ -functions in (36), we can therefore find the normalized three-point functions for Toda vertices $e^{\gamma\alpha^a \cdot \varphi}$ ($a = 1, \dots, r$) in closed form as

$$\begin{aligned} \langle e^{\gamma\alpha^a \cdot \varphi(z_1)} e^{\gamma\alpha^b \cdot \varphi(z_2)} e^{\gamma\alpha^c \cdot \varphi(z_3)} \rangle &= \frac{1}{A_0} G_3(z_1, z_2, z_3 | \gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c) \\ &= -\frac{\left(\frac{\mu^2}{8\pi\gamma^2} \right)^{-3}}{|z_{12}|^2 |z_{23}|^2 |z_{31}|^2} \times \begin{cases} s_0^a s_0^b s_0^c & (a \neq b \neq c) \\ s_0^a s_0^b (s_0^b - 1) & (a \neq b = c) \\ s_0^a (s_0^a - 1) (s_0^a - 2) & (a = b = c) \end{cases} \end{aligned} \quad (37)$$

The two-point couplings can also be related to the partition function for Toda vertices $e^{\gamma\alpha^a \cdot \varphi}$ ($a = 1, \dots, r$). Using (35) and (36), we find

$$A_0 = \left(\frac{\mu^2}{8\pi\gamma^2} \right)^2 \frac{A_2(\gamma\alpha^a)}{s_0^a (s_0^a - 1)}. \quad (38)$$

This gives the normalized two-point functions as

$$\begin{aligned} \langle e^{\gamma\alpha^a \cdot \varphi(z_1)} e^{\gamma\alpha^a \cdot \varphi(z_2)} \rangle &= \frac{1}{A_0} G_2(z_1, z_2 | \gamma\alpha^a, \gamma\alpha^a) \\ &= \left(\frac{\mu^2}{8\pi\gamma^2} \right)^{-2} \frac{s_0^a (s_0^a - 1)}{|z_{12}|^4}. \end{aligned} \quad (39)$$

Combining (36) and (38), we obtain the normalization free ratio among the couplings

$$\frac{(A_3(\gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c))^2 A_0}{A_2(\gamma\alpha^a)A_2(\gamma\alpha^b)A_2(\gamma\alpha^c)} = \begin{cases} \frac{s_0^a s_0^b s_0^c}{(s_0^a-1)(s_0^b-1)(s_0^c-1)} & (a \neq b \neq c) \\ \frac{s_0^a}{s_0^a-1} & (a \neq b = c) \\ \frac{(s_0^a-2)^2}{s_0^a(s_0^a-1)} & (a = b = c) \end{cases} \quad (40)$$

This may be useful in comparing the results obtained via different schemes with ours.

We finally note that the zero- and two-point fuctions obtained above are also consistent with the general relations

$$\begin{aligned} \left(\frac{\partial}{\partial\mu^2}\right)^3 A_0 &= -\frac{1}{V_0} \left(\frac{1}{8\pi\gamma^2}\right)^3 \sum_{a,b,c} \int d^2 z_1 d^2 z_2 d^2 z_3 G_3(z_1, z_2, z_3 | \gamma\alpha^a, \gamma\alpha^b, \gamma\alpha^c) , \\ \frac{\partial}{\partial\mu^2} G_2(z_1, z_2 | \beta, \beta) &= -\frac{1}{V_2} \frac{1}{8\pi\gamma^2} \sum_a \int d^2 z_3 G_3(z_1, z_2, z_3 | \beta, \beta, \gamma\alpha^a) , \end{aligned} \quad (41)$$

where V_0 denotes the infinite volume of the Möbius transformations given by

$$V_0 = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{|z_{12}|^2 |z_{23}|^2 |z_{31}|^2} . \quad (42)$$

We have found a kind of universality among the three-point coupling C_3 for vertex functions associated to the simple roots. The dependences on the simple roots appears only in the Γ -functions which arise in integrating over the zero-modes. This enables us to determine all the three-point vertex functions associated to simple roots up to an overall normarization. We may stress that all these results can be obtained essentially from the properties (26) without knowing the explicit form of the three-point couplings.

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